Behaviour in the Large of Numerical Solutions to One-Dimensional Nonlinear Viscoelasticity by Continuous Time Galerkin Methods[1]

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Abstract

We analyze the long time behaviour of fully discrete solutions to a one-dimensional nonlinear viscoelastic problem. It is shown that these approximations which are found by a continuous time Galerkin method converge to a steady state. The possible numerical steady states are characterized and in particular their high degree of dependence on initial data and mesh design is explained. Computational results are included which show the above dependence and indicate that the numerical solutions will typically not converge to unstable states.

1. Introduction

Recently, attempts have been made at minimizing nonconvex stored energy functionals by means of studying the steady states of auxiliary time dependent problems. In parts of the literature, the class of methods is termed Dynamic Relaxation. This technique involves solving a time dependent P.D.E. and marching to a steady state which hopefully is a near minimizer. We refer to [1], [2], [3], [4], [5], [6] and the references there. In this paper we will study the long term behaviour of certain fully discrete solutions to

$$
\begin{array}{ll}
U_{tt} = (\sigma(U_x) + U_{xt})_x & \text{in } (0,1) \times (0,\infty) \\
U(0,t) = 0, & t \geq 0 \\
(\sigma(U_x) + U_{xt})(1,t) \ \text{ or } \ U(1,t) = 0, & t \geq 0 \\
U(x,0) = U_0(x), \quad U_t(x,0) = V_0(x) & \text{in } (0,1)
\end{array}
\tag{1.1}
$$

which commands some interest in its own right. (1.1) models the one-dimensional motion under zero body forces of a nonlinear viscoelastic material of rate type, sometimes called the Kelvin-Voigt model. $U(x,t)$ denotes the displacement at time t of a particle having position x in some reference configuration. See also [7]. It is hoped then that for t sufficiently large, U will approach a local minimum and satisfy the Euler-Lagrange equations associated with the minimization problem.

The asymptotic behaviour of (1.1) was the focus of two studies, [8] and [9], in which it was shown that weak solutions exist globally provided $U_0 \in W^{1,\infty}$, $V_0 \in L^2(0,1)$ and the sign of the stresses for large s is restricted as follows: $\sigma : R \rightarrow R$,

$$
\exists M > 0, \quad |\, s \,| > M \Longrightarrow s\,\sigma(s) > 0. \tag{1.2}
$$

Furthermore the solutions converge strongly to (local) equilibrium, as $t \rightarrow \infty$:

$$
\begin{array}{l}
U_t \longrightarrow 0, \ \text{ in } \ H^1(0,1), \quad \sigma(U_x) + U_{xt} \longrightarrow 0 \ \text{ in } \ H^2(0,1) \\
U_x(x,t) \rightarrow S_\infty(x) \text{ boundedly a.e.}, \quad \sigma(S_\infty) = 0 \text{ a.e.,}
\end{array}
\tag{1.3}
$$

and when a dynamic stability criterion

$$
\frac{d\sigma}{ds}(\tilde{s}) \geq \sigma_0 > 0 \quad \text{for } \ \sigma(\tilde{s}) = 0 \tag{1.4}
$$

is satisfied, the asymptotic states are metastable in the sense that if the strains are perturbed by small amounts except on a set of measure ϵ, then the solution will approach an equilibrium with strain equal to the unperturbed strain everywhere but possibly the same exceptional set. $U(\cdot,t)$ converges in the sense of generalized curves, cf. [10] and [11], i.e. there exists a parametrized family of probability measures $\{\nu_x\}_{x\in(0,1)}$ on $\mathbf{R}$ such that $g(u_x(\cdot,t)) \xrightarrow{*} <\nu_x, g>$ in $L^\infty(0,1)$ for each continuous g. Here, $<\nu, g(s)>$ denotes the value of a probability measure ν for a continuous function g on the state domain.

It has been observed, e.g. by Silling [3], that dynamic relaxation as a computational method for a similar two-dimensional problem may yield numerical solutions exhibiting different phases that can be identified with appropriate ranges of values of the deformation gradient. In mixing regions between two elliptic phases, the gradients become discontinuous and the solution was observed to be highly mesh dependent.

We are here interested in studying (1.1) when σ is not a monotone increasing function so that the *stored-energy functional*

$$I_\psi(v) = \int_0^1 \psi(v')dx \tag{1.5}$$

is not convex. In general there are thus infinitely many solutions to the corresponding *equilibrium* problem:

$$\{\sigma(u'(x))\}' = 0 \tag{1.6}$$

subject to mixed, Dirichlet and traction boundary conditions

$$v(0) = f_0 \quad \text{and} \quad \sigma(v')(1) = P \tag{1.7a}$$

or Dirichlet boundary conditions

$$v(0) = f_0 \quad \text{and} \quad v(1) = f_1 \tag{1.7b}$$

ψ is a real-valued function in one variable: $\psi(v')(x) = \int_0^{v'} \sigma(s)ds$ and is defined as follows. Let $s_L < \bar{s} < s_U$, $0 < \bar{\alpha} < (s_U - s_L)/2$ and λ_1, λ_2, $\lambda_3 > 0$. We take $\bar{s} = f_1 - f_0 = P = 0$. Let ψ be a double well:

$$\begin{array}{lll}
\psi(s) > \psi(s_L) = \psi(s_U) = 0 & \text{for} & s \neq s_L, s_U, \\
\psi(s) \geq \lambda_1(s - s_L)^2 & \text{for} & \mid s - s_L \mid \leq \bar{\alpha}, \\
\psi(s) \geq \lambda_1(s - s_U)^2 & \text{for} & \mid s - s_U \mid \leq \bar{\alpha}, \\
\psi(s) \geq \lambda_1\bar{\alpha}^2 & \text{for} & s_L + \bar{\alpha} < s < s_U - \bar{\alpha}, \\
\lambda_3(s - \bar{s})^2 \geq \psi(s) \geq \lambda_2(s - \bar{s})^2 & \text{for} & s \notin [s_L - \bar{\alpha}, s_U + \bar{\alpha}].
\end{array} \tag{1.8}$$

I_ψ is well defined on H^1. Collins, Kinderlehrer, and Luskin [12] noted that the variational principle $\inf_{v\in H^1} I_\psi(v)$ where v satisfies (1.7b) may have nonunique limits for minimizing sequences as well as nonunique associated Young measure (the probability measures mentioned above), [11]. The simplest limit deformation, however, is the linear function f satisfying (1.7b). In [12] they consider instead minimizing

$$J_\psi(v) = I_\psi(v) + \int_0^1 (v(x) - f(x))^2 dx \tag{1.9}$$

Figure 1: Energy and Stress as functions of Strain

subject to $v(0) = v(1) = 0$ with f the unique limit deformation and a unique associated Young measure

$$\nu_x = \gamma\delta_{s_L} + (1-\gamma)\delta_{s_U} \tag{1.10}$$

where $\gamma = (s_U - \bar{s})/(s_U - s_L)$ and δ_z denotes the Dirac delta distribution with support at z. Note that $\inf_{H^1 \ni v:(1.7)} I_\psi(v) = \inf_{v\in H^1} J_\psi(v)$. The stored energy and the associated stress-strain law are be depicted in fig. 1 ($\bar{s} = 0$)

Note that, although $\inf_{v\in H^1} J_\psi(v) = 0$, there is no minimizer in H^1. In [12] it was shown that minimizing J_ψ over a sequence of (uniform mesh) finite element spaces, leads to a minimizing sequence u_h, such that u_h' converges in a weak sense to the unique Young measure in (1.10) as $h \to 0$. $u_h'(x)$ oscillates (in the limit) between the energy wells at $s = s_L$ and at $s = s_L$ occupying these states in proportions γ and $1-\gamma$ of the interval, respectively. Double wells of unequal heights can be incorporated as in [12] by shifting ψ by some linear function.

Example 1.1 Let us characterize, for future use, the solution to the discrete Galerkin, Euler-Lagrange equations in the simplest case. Let $m \in N$, for $h = \frac{1}{m}$, $x_0 = 0$ and $x_i = ih$, $I_i = (x_{i-1}, x_i)$ for $i = 1, \cdots, m$. Define the finite element space for the boundary conditions (1.7a) ($f_0 = \bar{s} = 0$, $I = (0,1)$)

$$S_h^1 = \{v \in C(0,1) : \quad v(0) = 0 \quad \text{and} \quad v\,|_{I_i} \in P_1(I_i), \;\; i = 1, \cdots, m\} \tag{1.11}$$

Then the weak form of the Euler-Lagrange equation $(\sigma(u_x))_x = 0$ becomes

$$\text{Find } u_h \in S_h^1 \text{ such that for all } v \in S_h^1 : \int_I \sigma(u_{h,x}) v_x \, dx = 0. \tag{1.12}$$

Since $u_h(0) = 0$ and $u_{h,x}$, v_x are piecewise constants, we can substitute $\sum_i \sigma(u_{h,x})(v(x_i) - v(x_{i-1}))$ for $\int_I$. Now successively testing against m basis functions $\hat{v}^i \in S_h^1$ defined by

$\hat{v}^i(x_i) = 1$, $\hat{v}^i(x_j) = 0$ for $j \neq i$, $i = 1, \cdots, m$, one easily see that $\sigma(u_h')$ remains the same constant throughout I. So $u_h \in C_h^1$ where

$$C_h^1 = \{v \in S_h^1 : \sigma(v_x) = 0\} = \{v \in S_h^1 : v_x \in \{s_L, 0, s_U\}\}, \tag{1.13}$$

such that C_h^1 consists of 3^m isolated solutions. Any two nonidentical solutions in C_h differ by at least a positive constant depending on h, s_L, and s_U.

Example 1.2 In the case of Dirichlet boundary conditions at both endpoints (1.7b), the discrete equilibria become less tangible. With m, h, x_i and I_i as in the previous example, define

$$S_{h,0}^1 = \{v \in C(0,1) : \quad v(0) = v(1) = 0, \; v\,|_{I_i} \in P_1(I_i), \; i = 1, \cdots, m\} \tag{1.14}$$

as above we see that $\sigma(u_{h,x})$ is constant P, say. Let $\sigma^{-1}(P) = \{\mu_1, \mu_2, \mu_3\}$ listed in ascending order. Let m_i denote the numbers of intervals in which $u_{h,x} = \mu_i$, $i = 1, 2, 3$. Since u_h satisfies (1.7b) we get the following constraint

$$\sum_{i=1}^{3} m_i \mu_i = 0, \quad m_i \geq 0 \text{ for } i = 1, 2, 3.$$

The set of possible values for P depends very much on the shape of the stress strain law. It is clear that $\sigma^{-1}(P) = \{\mu_1\}$ a singleton is not viable, similarly if $\sigma^{-1}(P) = \{\mu_1, \mu_2\}$, we need $\mu_1/\mu_2 \in Q$ and m sufficiently large. To explain take a very simple minded trilinear stress-strain law, such that σ has roots s_L, 0, and s_U; $|\sigma'| = \lambda$. Thus

$$\sigma(s) = \begin{cases} \lambda(s - s_L), & \text{for } s < s_L/2 \\ -\lambda s, & \text{for } s_L/2 \leq s \leq s_U/2 \\ \lambda(s - s_U), & \text{for } s > s_U/2 \end{cases}$$

In one of the two root cases, $P = -\lambda s_L/2$ and $\mu = s_L/2$, $\mu_2 = s_U - s_L/2$ so that s_L/s_U must be rational ($-\frac{p}{q}$, irreducibly say) and $m \geq p + q$. In the three root cases $-P/\lambda \in \frac{1}{2}(-s_U, -s_L)$ and $\mu_1 = s_L + P/\lambda$, $\mu_2 = -P/\lambda$, and $\mu_3 = s_U + P/\lambda$. the constraint of zero mean slope above can be met always by letting $m_2 = 0$ and choosing $P/\lambda = -(m_1 s_L + m_3 s_u)/m$. This allows for at least one degree of freedom in choice for P and thus possible slopes μ_i, as long as $m_i \geq m/4$, $i = 1, 3$. It is particularly interesting that we can allow $m_2 > 0$ putting strict lower bounds on m_1, m_3. The solutions are still isolated, the P/λ values admissible being separated by $d_h = c(s_L, s_u)/m$ amounting to a nearest neighbour distance of the same order in the set of discrete Galerkin equilibria. Of course, more nontrivial discrete equilibria are realizable using nonuniform meshes selected appropriately.

Increasing the polynomial degree in the finite element space will likely yield an even wider spectrum of discrete equilibria.

Other forms of damping/dissipation could also be considered for dynamic relaxation: thermal (heat diffusion), frictional (including U_t in (1.1)), and viscoelastic of history type ($\sigma(U_x)$ depending on the deformation for all previous times through convolution with a kernel (see Bielak and MacCamy [13]). Dissipation mechanisms which will substantially influence behaviour include capillarity (adding a U_{xxxx} term to (1.1), see Slemrod [15]) and non-local (in space) constitutive relations , see Belytschko and Bazant [5]. These are typically not used in dynamic relaxation with the exception of Belytschko.

Finally note, that if the energy in contrast is convex, a unique solution is given by the Euler-Langrange equations and there exist numerical procedures for minimization, see e.g. [16], and the time dependent P.D.E. was handled, see [17], [18], [19]. If one uses the Maxwell relation to relax the problems, nonuniqueness of an even larger class than before arises. Numerical methods exist, cf. [20] and the references there.

Our goal will be to establish the set of conditions under which a class of numerical methods will yield a long time behaviour with asymptotic states that are (local) minima of (1.9).

The plan of the paper is as follows. We analyzed what possible numerical steady states exist for (1.1) already. In the next section, we introduce the continuous time Galerkin scheme. In section 3, we show that the numerical solution must converge to one of the steady states (for the specific fully discrete method in section 2). We report on some of our numerical experiments in section 4 which is followed by some concluding remarks.

2. Dynamic Relaxation by Continuous Time Galerkin Schemes with Viscoelasticity

We introduce the continuous time Galerkin (CTG) schemes (see [22] and the references there). Let $0 = x_0 < x_1 < \ldots < x_{i-1} < x_i < \ldots < x_m = 1$, $I_i = (x_{i-1}, x_i)$, $h_i = | I_i |$, $i = 1, \cdots, m$ and

$$S_h^1 = \{\chi \in C(0,1) : \ \chi(0) = 0, \ \ \chi|_{I_i} \in P_1(I_i), \ \ i = 1, \cdots, m\}.$$

Let $0 = t_0 < t_1 < t_2 < \ldots$, $J_n = (t_{n-1}, t_n)$, $k_n = | J_n |$, $n = 0, 1, 2, \ldots$ and

$$S_k^1 = \{\tau \in C(0,\infty) : \ \ \tau|_{J_n} \in P_1(J_n)\}.$$

Then $S_{hk}^1 = S_h^1 \otimes S_k^1 = \{\lambda : \ \lambda = \sum_{j=1}^{l} \chi_j \tau_j, \ \chi_j \in S_h^1, \ \tau_j \in S_k^1, \ l \in N\}$ and we discretize as follows (CTG):

$$\left\{\begin{array}{l} \text{Find } u \in S_{hk}^1 \text{ and } v \in S_{hk}^1 \text{ such that} \\[2ex] ((u_t - v, \chi))_n = 0 \quad \forall \chi \in S_h^1 \otimes P_0(J_n) \\[2ex] ((v_t, \lambda))_n + ((u_{xt} + \sigma(u_x), \lambda_x))_n = 0 \quad \forall \lambda \in S_h^1 \otimes P_0(J_n) \\[2ex] \text{where } u_x(\cdot, 0) = u_{0,x} \cong U_{0,x} \text{ and } v(\cdot, 0) = v_0 \cong V_0. \end{array}\right. \tag{2.1}$$

The inner products are defined by

$$\begin{aligned} (v, w) &= \textstyle\int_0^1 v(x) w(x)\, dx \\ (v, w)_n &= \textstyle\int_{J_n} v(t) w(t)\, dt \\ ((v, w))_n &= \textstyle\int_0^1 \int_{J_n} v(x,t) w(x,t)\, dt dx. \end{aligned} \tag{2.2}$$

Note that (2.1) can be given the following equivalent finite difference in time formulation

$$\begin{aligned} &\textstyle\int_0^1 \left(\frac{u^{n+1} - u^n}{k} - \frac{v^{n+1} + v^n}{2} \right) \chi dx = 0, \quad \forall \chi \in S_h^1 \\ &\textstyle\int_0^1 \left(\frac{v^{n+1} - v^n}{k} \lambda + \left(\frac{u_x^{n+1} - u_x^n}{k} + \frac{\psi(u_x^{n+1}) - \psi(u_x^n)}{u_x^{n+1} - u_x^n} \right) \lambda_x \right) dx, \quad \forall \lambda \in S_h^1, \end{aligned} \tag{2.3}$$

where $w^j = w(t_j)$ for $j = n, n+1$; $w = u, u_x, v \in S_{hk}^1$. One of the fundamental properties is the following energy estimate.

Proposition 2.1 *Let $u, v \in S^1_{hk}$ be the solutions of (2.1). Then for any $n \in N$, the following energy identity holds*

$$\int_0^1 (\frac{1}{2}v^2 + \psi(u_x))\, dx \,|_{t_n}^{t_{n+1}} = -\int_0^1 \int_{J_n} u_{xt}^2 \, dt dx \tag{2.4}$$

Proof: Choose $\chi = v_t$ and $\lambda = u_t$ in (2.1) to get

$$((u_t, v_t))_n = \frac{1}{2}\int_0^1 v^2 \, dx \,|_{t_n}^{t_{n+1}},$$

and

$$((u_t, v_t))_n = -((\sigma(u_x) + u_{xt}, u_{xt}))_n = -\int_0^1 \psi(u_x)\, dx \,|_{t_n}^{t_{n+1}} - \int_0^1 \int_{J_n} u_{xt}^2 \, dt dx.$$

from which (2.4) follows.

□

Corollary 2.2 *Under the same hypothesis as in Prop. 2.1,*

$$E(t_n) = E_0 - \int_0^{t_n} \int_0^1 u_{xt}^2 \, dx dt \tag{2.5}$$

where

$$\begin{aligned} E(t_n) &= \textstyle\int_0^1 (\frac{1}{2}v^2 + \psi(u_x))(x, t_n)\, dx, \\ E_0 &= E(t_0). \end{aligned} \tag{2.6}$$

Proof: Sum (2.4) from $j = 0$ to $j = n - 1$.

□

We still have to prove existence and uniqueness of solutions to (2.1)

Lemma 2.3 *If (u, v) is a solution to (2.1), then there exists an $M > 0$, depending only on E_0, such that for all $n \in N$*

$$\begin{aligned} &(i) \quad \|v^n\|_{L^2(0,1)} \le M \\ &(ii) \quad \|\psi(u_x^n)\|_{L^1(0,1)} \le M \\ &(iii) \quad \|u^n\|_{H^1(0,1)} \le M. \end{aligned} \tag{2.7}$$

Proof: (2.5) yields $\|v^n\|_{L^2(0,1)} \leq (2E_0^{1/2})$ and $\|\psi(u_x^n)\|_{L^1(0,1)} \leq E_0$ from which (*i*) and (*ii*) follow. Using (*ii*) and subdividing I according to which of the last four sets defined in (1.8), u_x^n belongs, the L^2 norm of u_x^n is bounded and (*iii*) follows by Poincaré's inequality.

□

We then phrase (2.1) as a fixed point problem. Consider solving the CTG problem on the time slab $Q_n = (0,1) \times J_n$. Define the map $\Phi : (S_h^1 \otimes P_1(J_n))^2 \to (S_h^1 \otimes P_1(J_n))^2$ as the solution $(\tilde{u}^{j+1}, \tilde{v}^{j+1})$, given $(\tilde{u}^j, \tilde{v}^j)$, to

$$\begin{cases} \text{Find } \tilde{u}^{j+1}, \tilde{v}^{j+1} \in S_h^1 \otimes P_1(J_n) \text{ such that} \\ ((\tilde{u}_t^{j+1} - \tilde{v}^{j+1}, \chi))_n = 0, \quad \forall \chi \in S_h^1 \\ ((\tilde{v}_t^{j+1}, \lambda))_n + ((\tilde{u}_{xt}^{j+1}, \lambda_x))_n = -((\sigma(\tilde{u}_x^j), \lambda_x))_n, \quad \forall \lambda \in S_h^1 \\ \text{where } \tilde{u}_x^{j+1}(x,t) = u_x^n(x) \text{ and } \tilde{v}_x^{j+1}(x,t_n) = v^n(x), \quad \forall x \in (0,1). \end{cases} \tag{2.8}$$

This defines a linear system of equations. To show there is a unique solution on each iteration we prove the homogeneous system (zero right-hand-side and zero data at $t = t_n$) has only the zero solution.

Lemma 2.4 *Let Φ be defined as in (2.8). Then $\Phi^{-1}(0) = 0$.*

Proof: With zero right-hand-side in (2.8), let $\lambda = \tilde{u}_t^{j+1}$ and $\chi = \tilde{v}_t^{j+1}$ so that

$$\frac{1}{2}\int_0^1 (\tilde{v}^{j+1}(x, t_{n+1}))^2 dx + \int_{J_n}\int_0^1 \tilde{u}_{xt}^{j+1}\, dx dt = 0$$

and $\tilde{u}_{xt}^{j+1} \equiv 0$, $\tilde{u}^{j+1} \equiv 0$, and $\tilde{v}^{j+1}(t_{n+1}) \equiv 0$. Now choose $\lambda = \tilde{v}_t^{j+1}$ to get $((\tilde{v}_t^{j+1}, \tilde{v}_t^{j+1}))_n = 0$ and $\tilde{v}_t^{j+1} \equiv 0$ and $\tilde{v}^{j+1} \equiv 0$.

□

Let

$$B_R^n = \{\chi \in S_h^1 \otimes P_1(J_n) : \max_{t \in J_n} \|\chi_x(\cdot, t)\|_{L^2(0,1)} \leq R\}$$

and $M_\sigma = \sup_{|s| \leq M+1} |\sigma(s)|$. Then Φ selfmaps B_R^n into B_R^n for $R = M+1$, M being the constant in (2.7).

Lemma 2.5 *let Φ be defined as in (2.8). Then $\Phi : B_{M+1}^n \to B_{M+1}^n$, provided $k_n(M^2 + k_n M_\sigma^2) \leq 1$.*

Proof: As in the previous proof, let $\lambda = \tilde{u}_t^{j+1}$ and $\chi = \tilde{v}_t^{j+1}$ to get

$$\frac{1}{2}\int_0^1 (\tilde{v}^{j+1})^2 dx \,\Big|_{t_n}^{t_{n+1}} + ((\tilde{u}_{xt}^{j+1}, \tilde{u}_{xt}^{j+1}))_n = -((\sigma(\tilde{u}_x^j), \tilde{u}_{xt}^{j+1}))_n$$

or

$$\begin{aligned}\int_0^1 (\tilde{v}^{j+1})^2(x, t_{n+1})dx &+ \int_0^1 \int_{J_n} (\tilde{u}_{xt}^{j+1})^2 dt\,dx \\ &\le \int_0^1 (v^n)^2 dx + \int_0^1 \int_{J_n} (\sigma(\tilde{u}_x^j))^2 dx\,dt \\ &\le M^2 + k_n M_\sigma^2\end{aligned}$$

but

$$\|\tilde{u}_x^{j+1}(\cdot, t) - u_x^n\|_{L^2(0,1)} \le k_n^{1/2}\Big(\int_0^1 \int_{J_n} (\tilde{u}_{xt}^{j+1})^2 dt\,dx\Big)^{1/2}$$

so that

$$\|\tilde{u}_x^{j+1}(\cdot, t)\|_{L^2(0,1)} \le \|u_x^n\|_{L^2(0,1)} + \{k_n(M^2 + k_n M_\sigma^2)\}^{1/2} \le M + 1$$

ending the proof.

□

Finally, Φ is a contraction in some appropriate norm. Let $L_\sigma = \sup_{|s|\le M+1} | \sigma'(s)|$ and $|||v|||^2 = ((v_x, v_x))_n$.

Lemma 2.6 *Let Φ be defined as in (2.8). Then $\exists\, \kappa \in (0,1)$ such that*

$$|||\Phi_1(\tilde{u}_1, \tilde{v}_1) - \Phi_1(\tilde{u}_2, \tilde{v}_2)||| \le \kappa\, |||(\tilde{u}_1 - \tilde{u}_2||| \qquad (2.9)$$

and

$$\|[\Phi_2(\tilde{u}_1, \tilde{v}_1) - \Phi_2(\tilde{u}_2, \tilde{v}_2)](t_{n+1})\|_{L_2(0,1)} \le \kappa\, |||\tilde{u}_1 - \tilde{u}_2|||$$

provided $k_n^{1/2} L_\sigma < 1$, $\forall(\tilde{u}_i, \tilde{v}_i) \in B_{M+1}^n$, $i = 1, 2$.

Proof: Let $\delta = \tilde{u}_1^{j+1} - \tilde{u}_2^{j+1}$ and $\epsilon = \tilde{v}_1^{j+1} - \tilde{v}_2^{j+1}$. Then from $(2.8)_1-(2.8)_2$, we get

$$((\delta_t - \epsilon, \chi))_n = 0 \quad \forall \chi \in S_h^1$$

$$((\epsilon_t, \lambda))_n + ((\delta_{xt}, \lambda_x))_n = -((\sigma(\tilde{u}_{1,x}^j) - \sigma(\tilde{u}_{2,x}^j),\ \lambda_x))_n \quad \forall \lambda \in S_h^1$$

Letting $\chi = \epsilon_t$ and $\lambda = \delta_t$, we get

$$\begin{aligned}\frac{1}{2}\|\epsilon(\cdot, t_{n+1})\|^2_{L^2(0,1)} &+ |||\delta_{xt}||| \\ &\le L_\sigma\, |||\tilde{u}_{1,x}^j - \tilde{u}_{2,x}^j|||\ |||\delta_{xt}||| \\ &\le \frac{1}{2}|||\delta_{xt}||| + \frac{1}{2}L_\sigma\, |||\tilde{u}_{1,x}^j - \tilde{u}_{2,x}^j|||^2\end{aligned}$$

Since $|||\delta_x||| \leq k_n^{1/2} \, |||\delta_{xt}|||$, we get

$$|||\delta_{xt}||| \leq k_n^{1/2} L_\sigma \, |||\tilde{u}^j_{1,x} - \tilde{u}^j_{2,x}|||^2$$

yielding (2.9).

□

This establishes the existence and uniqueness of the solution to (2.1) by applying the contraction mapping principle to (2.8).

3. Large Time Behaviour of Numerical Solution

We now analyze the behaviour in the large of solutions to Continuous Time Galerkin numerical method introduced in Section 2.

If we again denote $u^n = u(t_n)$ (and similarly for v^n), we next state and prove the following properties of the sequence $\{(u^n, v^n)\}$.

Proposition 3.1 *Let (u^n, v^n) be the sequence uniquely determined by (2.3) and u the corresponding discrete function. Then $\exists M > 0$, such that*

$$\begin{aligned}
&(i) \quad \|v^n\|_{L^2(0,1)} \leq M, \quad \forall n \in N \\
&(ii) \quad \|\psi(u_x^n)\|_{L^1(0,1)} \leq M, \quad \forall n \in N \\
&(iii) \quad \|u^n\|_{H^1(0,1)} \leq M, \quad \forall n \in N \\
&(iv) \quad \int_0^\infty \int_0^1 | u_{xt} |^2 \, dxdt \leq E_0 \\
&(v) \quad \int_{J_n} \int_0^1 | u_{xt} |^2 \, dxdt \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.1}$$

Proof: (i) through (iii) merely restate lemma 2.3. Using (2.5) in the limit $n \to \infty$ yields (iv) which has (v) as an easy consequence.

□

Corollary 3.2 *let (u^n, v^n) satisfy (2.3) and (u, v) be the discrete solution. Then*

$$\begin{aligned}
&(i) \quad \|u_t\|_{L^2(0,1)}, \ \|u_{xt}\|_{L^2(0,1)} \to 0 \text{ as } t \to 0 \\
&(ii) \quad (\sigma(u_x^n), \lambda_x) \to 0, \quad \forall \lambda \in S_h^1 \text{ as } n \to \infty
\end{aligned} \tag{3.2}$$

Proof: (i) follows from $(3.1.v)$ and Poincaré's inequality. Since $v \in S_k^1$ for each x, v_t is a piecewise constant $(v^{n+1} - v^n)/k$. From (2.1) and (3.2 i) it follows that $\forall w \in S_h^1$

$$\int_0^1 \int_{J_n} vw \, dt \, dx \to 0 \text{ as } n \to \infty$$

or

$$\int_0^1 A^n w \, dx \to 0 \text{ as } n \to \infty$$

where $A^n = \{v^n + v^{n-1}\}/2$. Note that $v^{n+1} - v^{n-1} = 2(A^{n+1} - A^n)$. Summing $(2.1)_2$ we have

$$
\begin{aligned}
\vdash [((\sigma(u_x), \lambda))_n &+ ((\sigma(u_x), \lambda))_{n-1} \\
&= [((u_{xt}, \lambda_x))_n + ((u_{xt}, \lambda_x))_{n-1}] \\
&\quad + \int_0^1 \lambda(\int_{t_{n-1}}^{t_{n+1}} v_t \, dt) \, dx \\
&= \sum_{j=n}^{n+1} ((u_{xt}, \lambda_x))_j + 4 \int_0^1 (A^{n+1} - A^n)\lambda \, dx.
\end{aligned}
$$

From (3.2.i) and (3.3) we have

$$
\lim_{n \to \infty} \sum_{j=n}^{n+1} ((\sigma(u_x), \lambda_x))_j = 0.
$$

But

$$
\begin{aligned}
u_x(t) &= u_x^n + u_{xt}(t - t_n) \quad \text{for } t \in J_{n+1} \text{ and} \\
u_x(t) &= u_x^n + u_{xt}(t_n - t) \quad \text{for } t \in J_n.
\end{aligned}
$$

Thus

$$
\sigma(u_x) = \sigma(u_x^n) + R
$$

where

$$
\|R\|_{L^2(0,1)} \le L_\sigma k \|u_{xt}\|_{L^2(0,1)} \to 0 \text{ as } n \to \infty.
$$

So

$$
\sum_{j=n}^{n+1} ((\sigma(u_x), \lambda_x))_j = 2k(\sigma(u_x^n), \lambda_x) + \tilde{R}
$$

where $\tilde{R} \to 0$ as $n \to \infty$. Thus (ii) holds.

□

Thus, if $\{u^n\}$ or a subsequence tends to a limiting function, the latter must belong to C_h^1. To see that $\{u^n\}$ converges, we proceed in two steps.

Proposition 3.3 *Let $\{u^n\}$ be the sequence uniquely determined by (2.3). Then there exists $u^* \in C_h^1$ and $\mu > 0$ depending on k the time step such that*

$$
\|u^{n+1} - u^*\| \le \mu \|u^n - u^*\| \tag{3.3}
$$

where $\|v\| = \|v\|_{L^2(0,1)}$.

Proof: Let (u^n, v^n) denote the solution to (2.3) and let

$$\theta_v^n = v^n, \quad \theta_u^n = u^n - u^* \tag{3.4}$$

for some $u^* \in C_h^1$ to be determined. From (2.3)

$$(\theta_n^{n+1}, \lambda) + k\left(\frac{\psi(u_x^{n+1}) - \psi(u_x^n)}{u_x^{n+1} - u_x^n} + u_x^{n+1}, \lambda_x\right) = (\theta_v^n, \lambda) + (u_x^n, \lambda_x)$$

so that

$$(\theta_v^{n+1}, \lambda) + (-kD + \theta_{u,x}^{n+1}, \lambda_x) = (\theta_v^n, \lambda) + (\theta_{u,x}, \lambda_x) \tag{3.5}$$

where we denote $D = \sigma(u_x^*) - (\psi(u_x^{n+1}) - \psi(u_x^n))/(u_x^{n+1} - u_x^n)$. Note that $u^* \in C_h^1$ satisfies the discrete Euler-Lagrange equation. Choosing $\lambda = \theta_u^{n+1}$ in (3.5) yields

$$(\theta_v^{n+1}, \theta_u^{n+1}) - (\theta_v^n, \theta_u^{n+1}) + \|\theta_{u,x}^{n+1}\|^2 = (kD, \theta_{u,x}^{n-1}) + (\theta_{u,x}^n, \theta_{u,x}^{n+1}). \tag{3.6}$$

Likewise

$$\left(\frac{u^{n+1} - u^n}{k}, \chi\right) = \left(\frac{v^{n+1} + v^n}{2}, \chi\right)$$

which with $\chi = v^{n+1} - v^n$ becomes

$$\left(\frac{\theta_u^{n+1} - \theta_u^n}{k}, \theta_v^{n+1} - \theta_v^n\right) = \frac{1}{2}((\theta_v^{n+1})^2 - (\theta_v^n)^2, 1)$$

or

$$(\theta_v^{n+1}, \theta_u^{n+1}) - (\theta_v^n, \theta_u^{n+1}) = \frac{k}{2}\|\theta_v^{n+1}\|^2 + (\theta_u^n, \theta_v^{n+1}) - (\theta_u^n, \theta_v^n) - \frac{k}{2}\|\theta_v^{n+1}\|^2$$

which we substitute into (3.6) and get

$$\frac{k}{2}\|\theta_v^{n+1}\|^2 + \|\theta_{u,x}^{n+1}\|^2 =$$

$$k(D, \theta_{u,x}^{n+1}) + (\theta_{u,x}^n, \theta_{u,x}^{n+1}) \tag{3.7}$$

$$-(\theta_u^n, \theta_v^{n+1}) + (\theta_u^n, \theta_v^n) + \frac{k}{2}\|\theta_v^n)\|^2$$

$$\tag{3.8}$$

Expanding $\psi(u_x^{n+1})$ about u_x^n in a Taylor series; using the previously established bounds on u_x^n, u_x^{n+1}, as well as noting $|u_x^*| < \max\{|s_L|, |s_U|\}$; and assuming σ' is Lipschitz gives the following inequality:

$$|D| \leq C(|\theta_{u,x}^n| + |\theta_{u,x}^{n+1}|)$$

where C is a constant. Applying the Schwarz inequality, the elementary inequality $(2xy \leq \epsilon x^2 + \epsilon^{-1}y^2,\ x, y, \epsilon > 0)$, and Poincaré's inequality, we can absorb terms involving superscripts $n+1$ on the right hand side of (3.8) into similar terms on the left and we arrive at

$$k\|\theta_v^{n+1}\|^2 + 3\|\theta_u^{n+1}\|^2 \leq \mu(k)\ (k\|\theta_v^n\|^2 + 3\|\theta_u^n\|^2), \quad \forall n \in N$$

which is (3.3) and $1 < \mu(k) < Ck^{-1/2}$ independent of n.

□

Theorem 3.4 *Let $k < k^*$ and (u^n, v^n) be the uniquely determined sequences from (2.3). Then $\exists u^* \in C_h^1$,*

$$\lim_{n\to\infty}(u^n, v^n) = (u^*, 0) \tag{3.9}$$

Proof: We adapt an argument in [23]. From (3.1.*iii*), $\|u^n\|_{H^1(0,1)}$ is bounded and we may extract a subsequence $\{u^{n_j}\}$ converging to some $u^* \in S_h^1$. Because of (3.2.*i*), $\|u^{n+1} - u^n\| \to 0$ as $n \to \infty$. Then also $u^{n_j+1} \to u^*$ as $j \to \infty$. Then $(\psi(u_x^{n_j+1}) - \psi(u_x^{n_j}))/(u_x^{n_j+1} - u_x^{n_j}) \to 0$ as $j \to \infty$ and $(\sigma(u_x^*), \lambda_x) = 0$ following (3.2.*ii*) so that $u^* \in C_h^1$ and has to be one of the isolated points there. Let

$$B(u^*, \epsilon) = \{\chi \in S_h^1 : \ \|\chi - u^*\| < \epsilon\}$$

and pick $\delta < d_h/2$ (cf. Example 1.1) so that u^* is the unique element in $C_h^1 \cap B(u^*, 2\delta)$. Let $\{u^{n_j}\}$ be a subsequence of $\{u^n\}$ such that $\{u^{n_j}\} \subseteq B(u^*, \delta/\mu)$. Then

$$\{u^{n_j+1}\} \subseteq B(u^*, \delta)$$

according to (3.3). We then distinguish between two mutually exclusive cases:

Case 1: $\exists\{u^{n_k}\}_{k\in N} \subseteq B(u^*, \delta)\backslash B(u^*, \delta/\mu)$. Then this new subsequence must have a limit in $B(u^*, \delta)\backslash B(u^*, \delta/\mu)$ which contradicts that u^* is the only element in $B(u^*, 2\delta) \cap C_h^1$.
Case 2: $\neg(\exists\{u^{n_k}\}_{k\in N} \subseteq B(u^*, \delta)\backslash B(u^*, \delta/\mu))$. Then the entire sequence $\{u^n\}$ converges to u^*.

□

4. Numerical Results

In this section we compute the solutions obtained by the CTG method in order to clarify our results, demonstrate the instability present in equilibrium values μ where $\sigma'(\mu) < 0$, and show that the large time solutions are dependent on both the mesh and the initial data.

To implement the CTG scheme we eliminated the velocities from (2.3) to obtain the following three-step method: Find $\{u^n\}_{n=1}^{\infty} \subset S_h^1$ such that

$$\int_0^1 (\partial_t^2 u^n \lambda + [\frac{1}{2}(\tilde{\sigma}(u_x^n, u_x^{n+1}) + \tilde{\sigma}(u_x^{n-1}, u_x^n)) + \partial_t u_x^n]\lambda_x)\, dx = 0 \quad \forall \lambda \in S_h^1,$$

where

$$\partial_t^2 u^n = \frac{1}{k^2}(u^{n+1} - 2u^n + u^{n-1}),$$

$$\partial_t u^n = \frac{1}{2k}(u^{n+1} - u^{n-1}),$$

$$\tilde{\sigma}(u,v) = \frac{\psi(u) - \psi(v)}{u - v}, \quad \text{and} \quad \psi'(u) = u(u - s_L)(u - s_U).$$

We factored the numerator in $\tilde{\sigma}$ to eliminate $u - v$ from the denominator. We have assumed uniform time steps. To solve this nonlinear system on each time step we used the following fixed point iteration where we searched for $z^{(j)} \to u^{n+1}$:

$$\begin{aligned}
&\int_0^1 (z^{(j+1)}\lambda + \frac{k}{2} z_x^{(j+1)} \lambda_x)\, dx \\
&= \int_0^1 \{(2u^n - u^{n-1})\lambda + \frac{k}{2} u_x^{n-1}\lambda_x\}\, dx \\
&+ \frac{k^2}{2}\int_0^1 (\tilde{\sigma}(u_x^n, z_x^{(j)}) + \tilde{\sigma}(u_x^{n-1}, u_x^n))\lambda_x\, dx \quad \forall \lambda \in S_h^1
\end{aligned}$$

On each iteration the linear tridiagonal system of equations was solved by Gaussian elimination. Since our primary interest was in the behaviour of solutions of (2.3) we continued the iteration until $\|z^{(j+1)} - z^{(j)}\|_{L^\infty} \leq 10^{-8}$ rather than carry out a more efficient incomplete iteration procedure.

Figures 2, 3, and 4 show large time solutions (which we believe to be the final states for these problems). In all these cases we took the final time as $T = 40$, the number of time steps as $N = 200$, and initial data, specified completely at the nodes, as

$$u^0(x_j) = u^1(x_j) = 10^{-10}\sin(20x_j) \quad \text{for } j = 0, 1, \cdots, m.$$

We took $m = 9, 10, 20$ in Figures 2, 3, and 4, respectively. In all three cases we had

$$\|\sigma(u_x^n)\|_{L^\infty} \leq 5 \times 10^{-10}.$$

We conclude that each of the large time solutions is significantly different and none are close to the zero solution which does solve the Euler-Lagrange equation, but, is apparently dynamically unstable.

Figures 5, 6, and 7 show the approximation of the large time solutions of the problem with Dirichlet conditions at both endpoints. In each of these cases we took $T = 80$, $N = 400$, and the initial data

$$u^0(x) = 10^{-10}\sin(20\pi x_j) \quad \text{for } j = 0, 1, \ldots, m.$$

We took $m = 9, 10, 20$ in figures 4, 5, 6, respectively. In addition, we found $P = .2078265621 \pm 5 \times 10^{-10}$, $.323969547 \pm 5 \times 10^{-10}$, $0.0 \pm 5 \times 10^{-10}$ in the three cases.

We tried a larger number of timesteps in some of the cases above and found no change in the steady state patterns formed.

A possible explanation of the instability of the zero solution was observed in [8] by studying a linearized problem around zero. We do the same, as follows: Find $w = w(x,t)$ so

$$w_{tt} = \sigma'(0)w_{tt} + w_{xxt}$$

with

$$w(0,t) = w_x(1,t) = 0, \quad t > 0$$

and

$$w(x,0) = U_0(x), \quad w_t(x,0) = V_0(x) \quad \text{in } (0,1).$$

The solution has the form

$$w(x,t) = \sum_{n=1}^{\infty}(A_n e^{\lambda_n^+ t} + B_n e^{\lambda_n^- t})\sin((2n-1)\pi x)$$

where

$$\lambda_n^{\pm} = \frac{1}{2}(-(2n-1)^2\pi^2 \pm \sqrt{(2n-1)^4\pi^4 - 4\sigma'(0)(2n-1)^2\pi^2}).$$

Thus

$$e^{\lambda_n^+ t} \cong e^{-\sigma'(0)t}$$

$$e^{\lambda_n^- t} \cong e^{\sigma'(0) - n^2\pi^2} \to 0 \quad \text{as } n \to 0.$$

Thus the solution of the linearized system has a component which is growing exponentially.

A similar procedure, although more complicated, can be carried out on the fully discrete approximation problem to show the same results when $k = ch$ where c is a constant and h is sufficiently small.

5. Concluding Remarks

We have shown that numerical solutions to the model problem (1.7) found by the continuous time Galerkin scheme converge, as $t \to \infty$, to a numerical steady state, as one would wish. If this time-dependent process is viewed as dynamic relaxation, it is succesful at attaining steady states. We prove and observe though, as Silling [3] observed for a two dimensional anti-plane shear problem, that these are highly dependent on the mesh and initial data. We can also interpret our results as an analysis of the large time behaviour of the Kelvin-Voigt model for viscous damping using a general, systematic approach.

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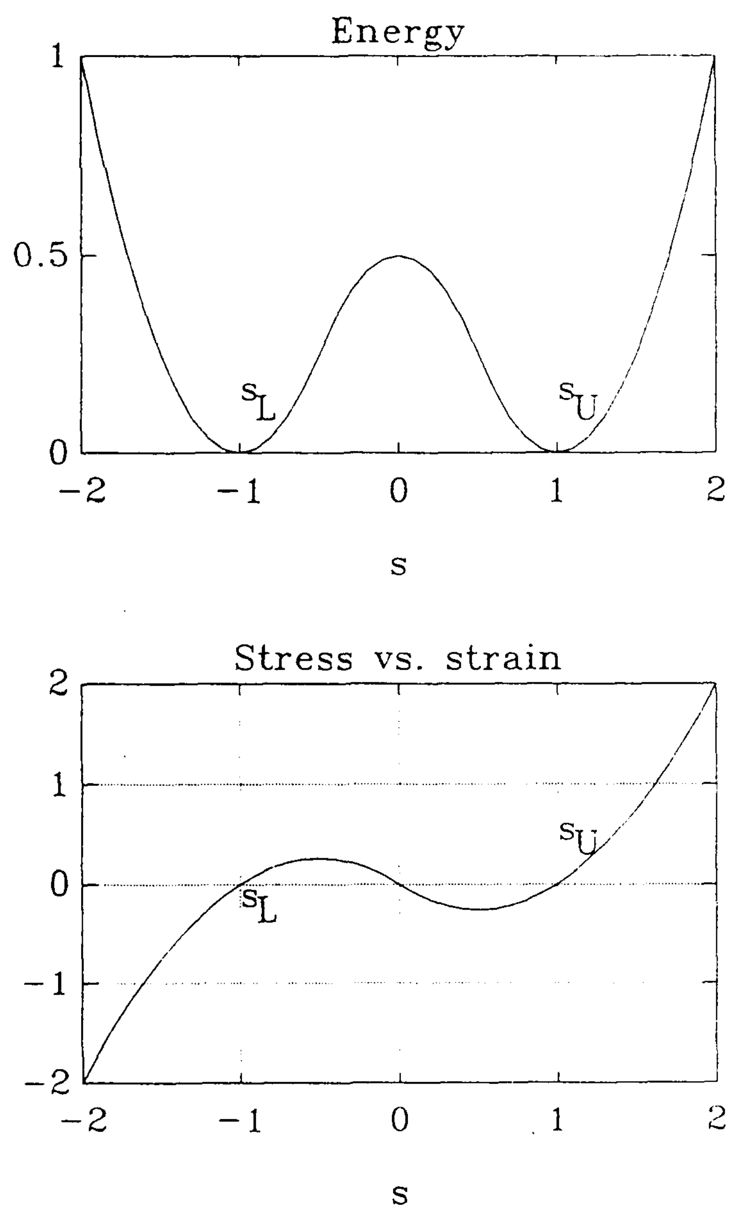

(fig. 1 p. 4)

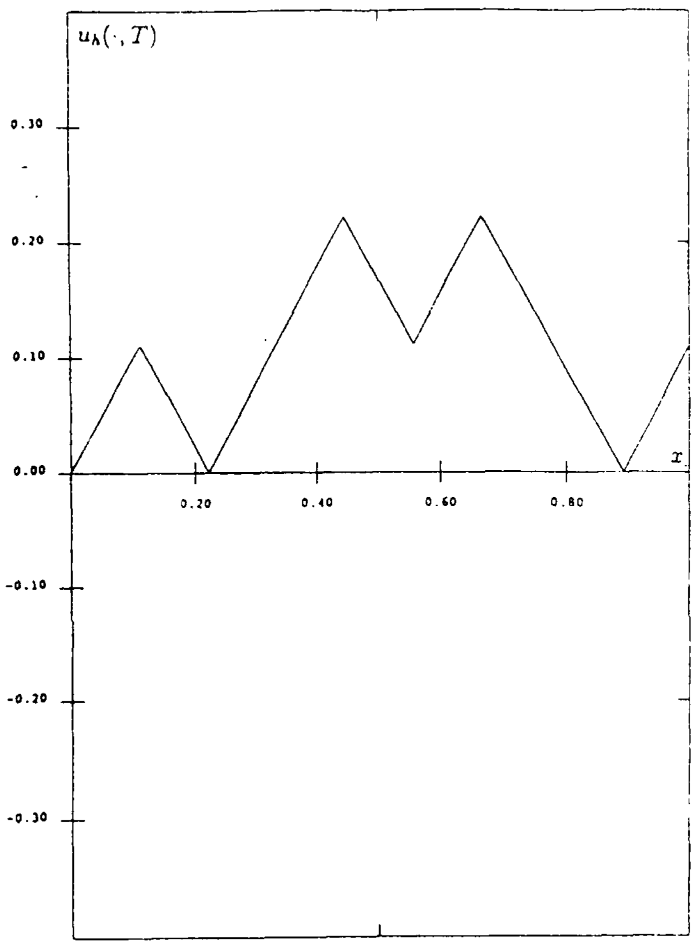

Figure 2 Approximate solution, u_h, to problem (1.7) with a Dirichlet condition at the left endpoint and traction condition at the right. There were 9 mesh intervals in space and a total of 200 timesteps were used.

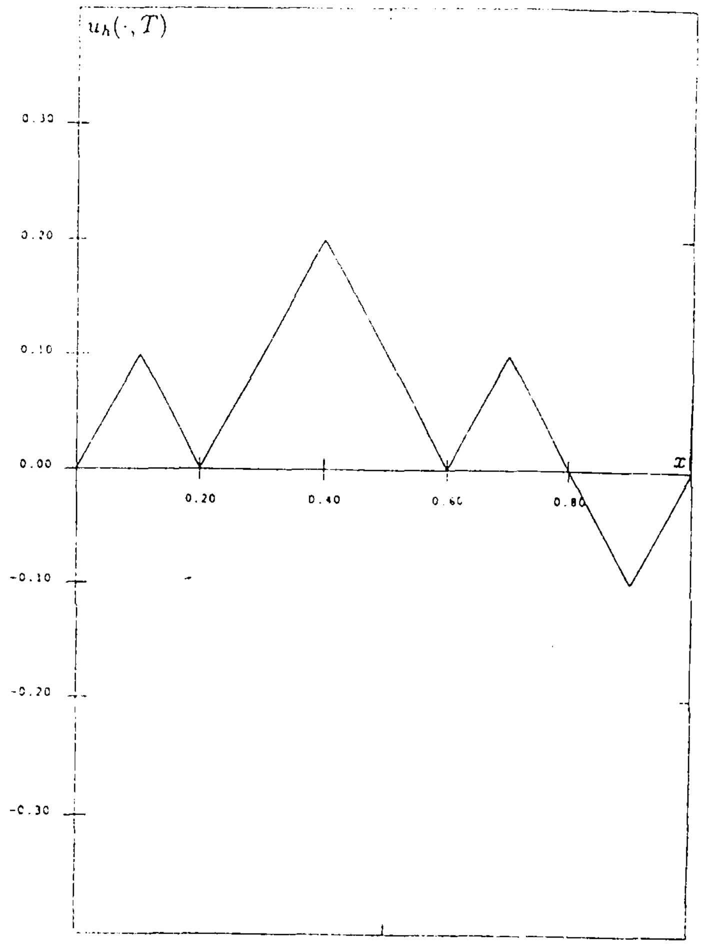

Figure 3 Approximate solution, u_h, to problem (1.7) with a Dirichlet condition at the left endpoint and traction condition at the right. There were 10 mesh intervals in space and a total of 200 timesteps were used.

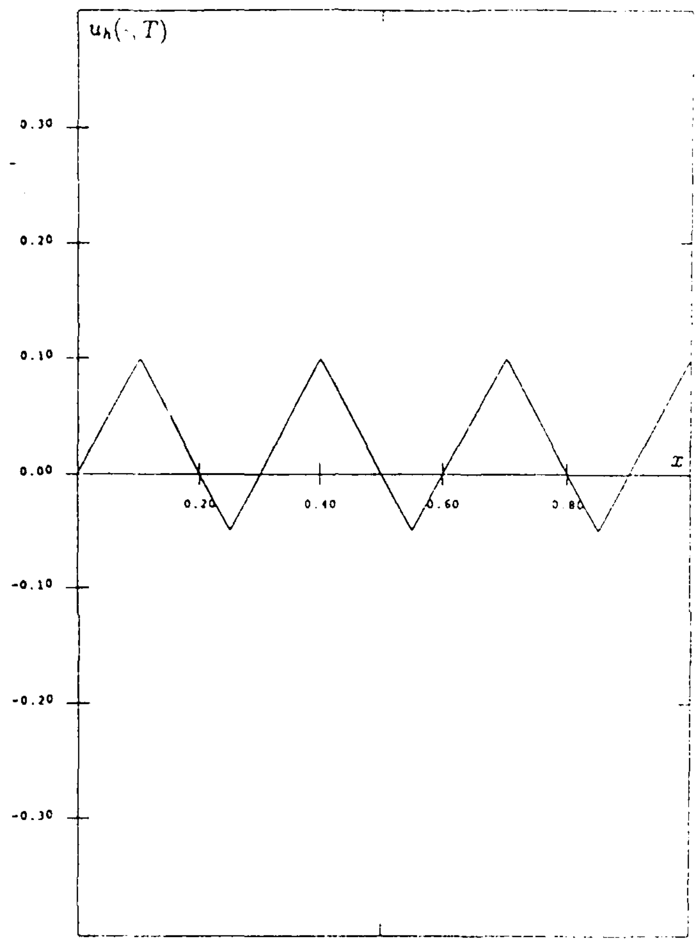

Figure 4 Approximate solution, u_h, to problem (1.7) with a Dirichlet condition at the left endpoint and traction condition at the right. There were 20 mesh intervals in space and a total of 200 timesteps were used.

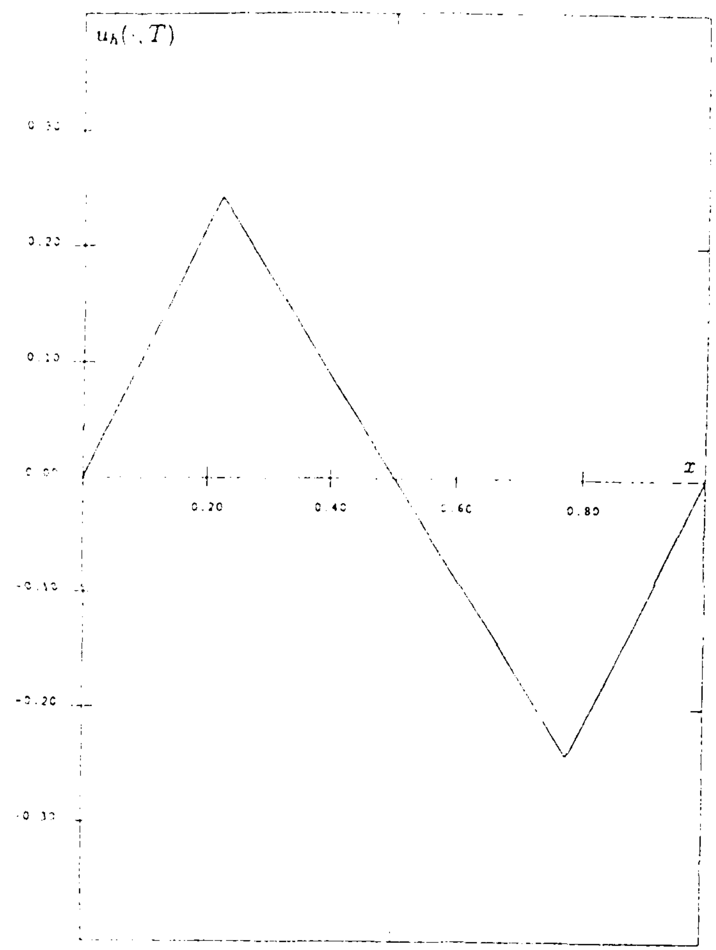

Figure 5 Approximate solution, u_h, to problem (1.7) with Dirichlet conditions at each endpoint. There were 9 mesh intervals in space and a total of 400 timesteps were used. $P = 2078$.

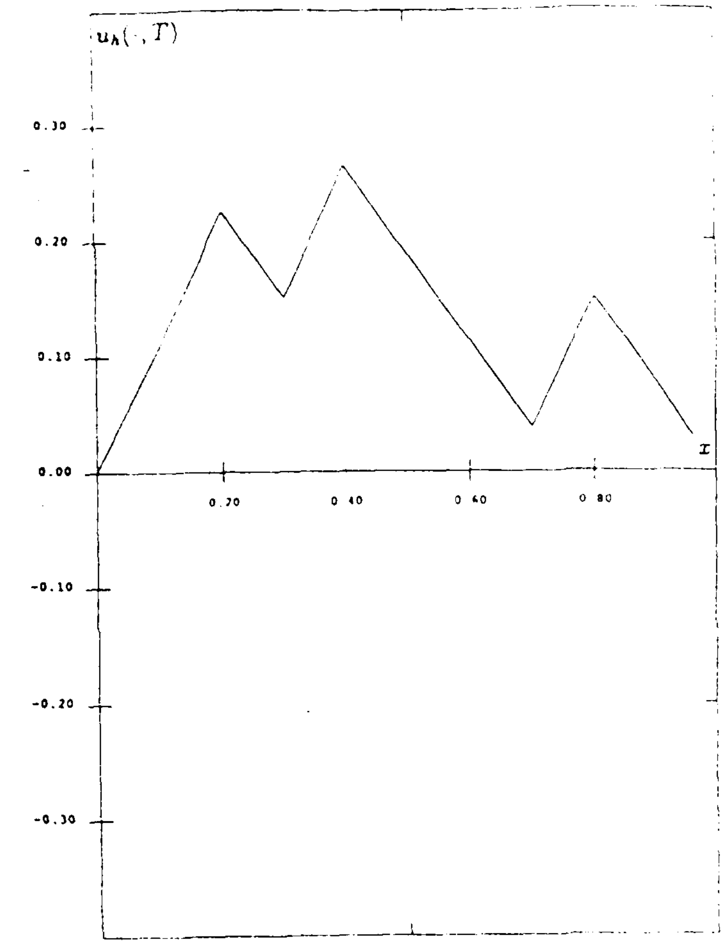

Figure 6 Approximate solution, u_h, to problem (1.7) with Dirichlet conditions at each endpoint. There were 10 mesh intervals in space and a total of 400 timesteps were used. P = .3240

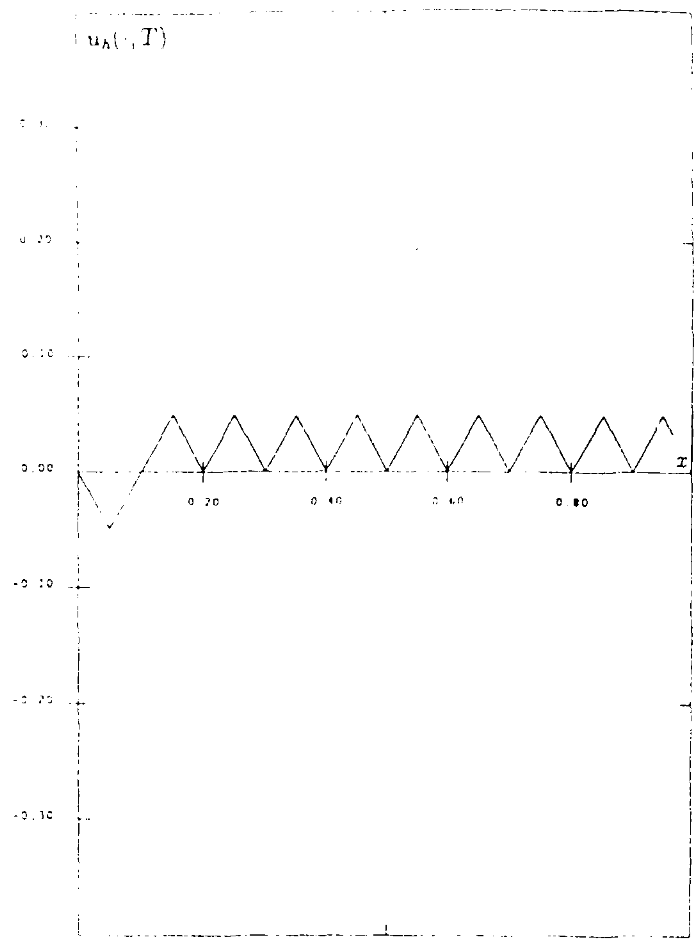

Figure 7 Approximate solution, u_h, to problem (1.7) with Dirichlet conditions at each endpoint. There were 20 mesh intervals in space and a total of 400 timesteps were used. P = 0.0.